Exactly solvable two-way traffic model with ordered sequential update

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Within the formalism of the matrix product ansatz, we study a two-species asymmetric exclusion process with backward and forward site-ordered sequential updates. This model, which was originally introduced with the random sequential update [J. Phys. A **30**, 8497 (1997)], describes a two-way traffic flow with a *dynamic impurity* and shows a phase transition between the free flow and the traffic jam. We investigate characteristics of this jamming and examine similarities and differences between our results and those with a random sequential update. [S1063-651X(99)06512-5]

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I. INTRODUCTION

The one-dimensional asymmetric simple exclusion process (ASEP) has been the subject of rigorous and intensive studies in recent years [1,2]. A variety of phenomena can be modeled by the ASEP and its generalizations (see Refs. [2,3], and references therein). The model has a natural interpretation as a description of traffic flow (Ref. [4], and references therein) and constitutes a basis for more realistic ones [5]. In traffic flow theories, the formation of traffic jams is one of the fundamental problems. Apart from their spontaneous formation [5], they can be also produced by hindrances such as road works or slow cars. Although these hindrances act locally, they may induce global and macroscopic effects on a system. This kind of behavior is one of the characteristic properties in nonequilibrium systems and has been studied in the context of driven lattice gases [6–12].

In ASEP models, two kinds of impurities are discussed in the literature. The first one is ''dynamic impurities,'' i.e., defective particles which jump with a rate lower than others [8,10,11,13]. In traffic terminology, such moving defects can be visualized as slow cars on a road, which in certain situations can induce a phase transition from the free flow to the congested flow. The other kind of impurities are ''static impurities,'' such as imperfect links where the hopping rate is lower than in other links [7,9,12,14–16]. Static impurities can also produce shocks in a system [7,9]. For either type of impurities, a limited amount of exact results is available and most of them are for models with the random sequential update [8,10,11,14]. For the fully parallel update, which is most realistic in traffic flow problems, exact solution are very rare [5,13,17,18], and most studies instead utilize approximation methods or numerical approaches [16,19,20].

Recently a two-way traffic model was introduced [11] where cars move forward in one lane and trucks move backward in the other lane, and both cars and trucks reduce their speeds when they approach each other. Within the matrix product ansatz (MPA) formalism, a modified version of this model is solved exactly in the particular case when there is only one truck in the system. This truck behaves as a dynamic impurity and induces a phase transition between the free flow and the congested flow of the cars. Various characteristics of the phase transition are examined.

In this paper we study an exactly solvable traffic model with two types of ordered sequential updates, which is identical to the model studied in Ref. [11] except for the updating schemes. More precisely, the updating schemes that we consider are backward and forward site-ordered sequential updating schemes in which one updates links of the chain sequentially. Alternatively one may use the so-called particle-ordered sequential updating scheme in which one sequentially updates the positions of the particles [13]. In this paper we restrict ourselves to the site-ordered sequential updates.

Presently it is of prime interest to determine whether distinct updating schemes can produce different behaviors. The implementation of the type of update is an essential part of the definition of a model, and some characteristics of the model may change dramatically. The aim of this work is to investigate consequences of changing the updating scheme of the model. Then, combined with the results in the random sequential updating scheme [11], we examine similarities and differences between the results in different types of updates. Our approach utilizes a mapping between the quadratic algebras of the ordered and random sequential updates, which has been initiated in the context of the one-species ASEP [21,22].

This paper has the following organization. In Sec. II we define the model, construct our MPA with both backward and forward sequential updates, and obtain the relevant quadratic algebras. Section III presents expressions of the average velocities of cars and the single truck, their thermodynamic limits, and a comparison with the corresponding results in Ref. [11]. In Sec. IV, we consider density profiles of cars and compute the probability to find a car at a distance x from the truck, which in the high density phase appears as a shock. We discuss the ambiguity in specifying the density profile, which is related to the nature of the updates, and introduce a definition of the averages to avoid the ambiguity. Section V is devoted to the density-density correla-

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tion function. In Sec. VI, we study the model in the presence of two trucks and evaluate the probability of their distance being R. The paper ends with some concluding remarks in Sec. VII.

II. MODEL DEFINITIONS AND MATRIX PRODUCT ANSATZ

A. Two-way traffic model with random sequential update

Here we describe briefly the two-way traffic model with the random sequential update (RSU) introduced in Ref. [11]. Consider two parallel one-dimensional chains, each with N sites. The periodic boundary condition applies to each chain. There are M cars and K trucks in the first and second chains, respectively. Cars move to the right and trucks move to the left. The state of the system is characterized by two sets of occupation numbers $(\tau_1,\tau_2,\ldots,\tau_N)$ and $(\sigma_1,\sigma_2,\ldots,\sigma_N)$ for the first and second lanes. If the site i of the car lane is occupied by a car, $\tau_i=1$, and zero otherwise. Similarly $\sigma_i=1$ if the site i of the truck lane is occupied by a truck, and $\sigma_i=0$ if the site is empty. In an infinitesimal time interval dt, a car (truck) hops to its right (left) empty site with the probability dt (γdt) if there is no truck (car) in front, and with the probability reduced by a factor β otherwise. More explicitly one has

$$(\tau_i,\tau_{i+1})=(1,0)\rightarrow(0,1)\quad \text{with rate} \begin{cases} 1 & \text{if } \sigma_{i+1}=0 \\ \dfrac{1}{\beta} & \text{if } \sigma_{i+1}=1 \quad (\text{truck in front}), \end{cases} \tag{1}$$

$$(\sigma_i,\sigma_{i+1})=(0,1)\rightarrow(1,0)\quad \text{with rate} \begin{cases} \gamma & \text{if } \tau_i=0 \\ \dfrac{\gamma}{\beta} & \text{if } \tau_i=1 \quad (\text{car in front}). \end{cases} \tag{2}$$

The reduction factor β, which ranges between 1 and infinity, is related to the width of roads: $\beta=1$ corresponds to a very wide road or a highway with a lane divider, and $\beta=\infty$ corresponds to a one lane road. Simulations with finite densities of cars and trucks show [11] that, in the steady state, the average velocities of cars and trucks decrease smoothly with increasing β. As an interesting limiting case, situations with a single truck is considered while the density of cars is kept finite. For this particular case, simulations suggest that for a given density n of cars, there exists a density-dependent critical value β_c, below which the average velocity of cars remains constant, and above which the average velocity decreases linearly with respect to $1-(1/\beta)$, a measure of the road narrowness. For $\beta>\beta_c$, the simulation also finds phase segregation into high (traffic jam) and low (free flow) density regions.

To investigate the characteristics of this single truck case analytically, the above two-lane model has been modified to an exactly solvable one. If one forbids a car and a truck to occupy two parallel sites i simultaneously, one can describe configurations with a single set of occupation numbers $\{\tau_i\}$ where $\tau_i=0$ (empty site), 1 (occupied by a car), or 2 (occupied by a truck). The following rules describe the modified dynamics:

$$\begin{aligned} &(1,0)\rightarrow(0,1) \quad \text{with rate } 1,\\ &(0,2)\rightarrow(2,0) \quad \text{with rate } \gamma,\\ &(1,2)\rightarrow(2,1) \quad \text{with rate } \frac{1}{\beta}. \end{aligned} \tag{3}$$

This model is equivalent to a two-species ASEP and can be solved exactly by the method of the matrix product state (MPS). The steady state weight P_s of a given configuration $(\tau_1,\tau_2,\ldots,\tau_N)$ is proportional to the trace of the normal product of some matrices:

$$P_s(\tau_1,\tau_2,\ldots,\tau_N)\sim \mathrm{Tr}(X_1X_2,\ldots,X_N) \tag{4}$$

where

$$X_i=\begin{cases} D & \text{for } \tau_i=1 \\ E & \text{for } \tau_i=2 \\ A & \text{for } \tau_i=0, \end{cases} \tag{5}$$

and these matrices satisfy the quadratic algebra

$$DE=D+E,\quad \alpha AE=A,\quad \beta DA=A \quad (\alpha\equiv\beta\gamma). \tag{6}$$

With the help of the MPS method, many important characteristics such as average velocities, density profiles, and k point correlation functions were obtained exactly in Ref. [11].

B. Two-way traffic model with ordered sequential updates

In the ordered sequential updating (OSU) schemes, time is discrete and the update occurs in an ordered way. To describe the dynamics in this scheme, it is convenient to introduce a Hilbert space that is spanned by the orthonormal ket vectors $|\{\tau\}\rangle=|\tau_1\rangle\otimes|\tau_2\rangle\otimes\ldots\otimes|\tau_N\rangle$. The state of a system at the jth time step can be represented by a ket vector $|P,j\rangle$,

$$|P,j\rangle=\sum_{\text{conf}} P(\{\tau\};j)|\{\tau\}\rangle, \tag{7}$$

where $P(\{\tau\};j)$ is the weight of the configuration $\{\tau\}$ at the jth time step, and Σ_{conf} denotes the summation over all possible configurations. The state at the next time step is then obtained from $|P,j\rangle$ by applying a transfer matrix

$$|P,j+1\rangle = T|P,j\rangle, \tag{8}$$

where the transfer matrix T takes a different form depending on the precise nature of the updates. In the backward sequential updating (BSU) scheme, for example, the transfer matrix T becomes $T_{\leftarrow}$, where

$$T_{\leftarrow} = T_{N,1}T_{1,2}\ldots T_{N-2,N-1}T_{N-1,N} \tag{9}$$

and in the forward sequential updating (FSU) scheme, it becomes $T_{\rightarrow}$ where

$$T_{\rightarrow} = T_{N,N-1}T_{N-1,N-2},\ldots,T_{2,1}T_{1,N}. \tag{10}$$

Each element in the products is defined by

$$T_{i,i+1} = T_{i+1,i} = \underbrace{\mathbf{1}\otimes\cdots\otimes\mathbf{1}}_{i-1}\otimes\mathcal{T}\otimes\underbrace{\mathbf{1}\otimes\cdots\otimes\mathbf{1}}_{N-i-1}. \tag{11}$$

Here $\mathbf{1}$ is the identity matrix acting on a local ket vector $|\tau\rangle$ and the local transfer matrix $\mathcal{T}$ acts on the tensor product state of two local ket vectors, $|\tau\rangle\otimes|\tau'\rangle$. In Eqs. (9) and (10), the site N is chosen as a starting point of the update for definiteness.

The local transfer matrix $\mathcal{T}$ varies depending on possible exchange processes allowed in a model. As a straightforward generalization of the two-way traffic model in Ref. [11], we allow the following processes to occur in each discrete time step:

$$\begin{aligned} &(1,0)\to(0,1) \quad \text{with probability} \quad \eta, \\ &(0,2)\to(2,0) \quad \text{with probability} \quad \eta\gamma, \\ &(1,2)\to(2,1) \quad \text{with probability} \quad \frac{\eta}{\beta}, \end{aligned} \tag{12}$$

where $0\leq\eta,\eta\gamma\leq 1\leq\beta$. For this two-way traffic model, off-diagonal matrix elements of $\mathcal{T}$ have nonzero values only for the elements

$$\begin{aligned} (\langle 0|\otimes\langle 1|)\mathcal{T}\,(|1\rangle\otimes|0\rangle) &= \eta, \\ (\langle 2|\otimes\langle 0|)\mathcal{T}\,(|0\rangle\otimes|2\rangle) &= \eta\gamma, \\ (\langle 2|\otimes\langle 1|)\mathcal{T}\,(|1\rangle\otimes|2\rangle) &= \frac{\eta}{\beta}, \end{aligned} \tag{13}$$

and the diagonal elements can be specified from the condition of the probability conservation, $\Sigma_{\tau_3,\tau_4}(\langle\tau_3|\otimes\langle\tau_4|)\mathcal{T}(|\tau_1\rangle\otimes|\tau_2\rangle)=1$, which ensures that the sum of entries in each column is equal to 1.

C. Matrix product state

Recently Krebs and Sandow [23] showed that the steady state of any stochastic process with arbitrary nearest neighbor interactions, defined on an open one dimensional systems with RSU, can be always written as a matrix product state. This theorem was soon generalized by Rajewsky and Schreckenberg [24] to stochastic processes with ordered sequential and sub-lattice parallel updates. For closed systems with the periodic boundary condition, on the other hand, such a theorem is not available at present and it is not yet clear how widely the MPA is applicable. Nevertheless, numerous recent studies [8,10,11,22,25–27] have proven that the MPA can be a powerful investigation tool even for closed systems as well. Motivated by recent successes, here we assume that the stationary state of the two-way traffic model can be written in terms of the MPS,

$$P_s(\tau_1,\tau_2,\ldots,\tau_N)\sim \mathrm{Tr}(X_1X_2,\ldots,\hat{X}_N) \tag{14}$$

where $X_i = D,\ E,$ or A ($\hat{X}_N=\hat{D},\ \hat{E},$ or $\hat{A}$) depending on τ_i (τ_N). Note that the matrices at the site N are different from those at other sites, which stems from the special role of the site N as a starting point of the update.[1] The necessity of the hatted matrices in the OSU scheme is clearly explained in Ref. [22].

By introducing two ket vectors $|U\rangle$ and $|\hat{U}\rangle$,

$$\begin{aligned} |U\rangle &= A|0\rangle + D|1\rangle + E|2\rangle, \\ |\hat{U}\rangle &= \hat{A}|0\rangle + \hat{D}|1\rangle + \hat{E}|2\rangle, \end{aligned} \tag{15}$$

where the coefficients $A,\ D,\ E,\ \hat{A},\ \hat{D}$, and $\hat{E}$ are matrices defined on a different auxiliary space, we can write Eq. (14) formally as

$$|P_s\rangle = \frac{1}{Z(N,M)}\mathrm{Tr}(|U\rangle\otimes|U\rangle\otimes\cdots\otimes|\hat{U}\rangle), \tag{16}$$

in which the normalization constant $Z(N,M)$ is the sum of all weights,

$$Z(N,M) = \sum_{\mathrm{conf}} \mathrm{Tr}(X_1,\ldots,\hat{X}_N), \tag{17}$$

and the trace is taken only over the normal product of the matrices $A,\ D,\ E,\ \hat{A},\ \hat{D}$, and $\hat{E}$.

By definition, $|P_s\rangle$ should be stationary under the action of the transfer matrix T

$$T|P_s\rangle = |P_s\rangle.$$

In the BSU scheme, the stationarity is guaranteed if the following relation holds:

$$\mathcal{T}\,(|U\rangle\otimes|\hat{U}\rangle) = |\hat{U}\rangle\otimes|U\rangle, \tag{18}$$

[1](The hatted-matrix in Eq. (14) breaks the translational invariance of the problem. As a result, the steady state weights are not completely fixed by the relative distances between cars and trucks. This situation is in contrast to Ref. [11] where the weights depend on the relative distances only due to the translational invariance [see Eq. (4)].)

which simply implies that upon the action of $T_{\leftarrow}$, the "defect" $|\hat{U}\rangle$ is transferred backward through the chain and returns back to the site N, its initial position. Equation (18) then leads to the following quadratic algebra:

$$[A,\hat{A}]=0, \quad [D,\hat{D}]=0, \quad [E,\hat{E}]=0,$$

$$A\hat{D}+\eta D\hat{A}=\hat{A}D, \quad (1-\eta)D\hat{A}=\hat{D}A,$$

$$\eta\gamma A\hat{E}+E\hat{A}=\hat{E}A, \quad (1-\eta\gamma)A\hat{E}=\hat{A}E,$$

$$\frac{\eta}{\beta}D\hat{E}+E\hat{D}=\hat{E}D, \quad \left(1-\frac{\eta}{\beta}\right)D\hat{E}=\hat{D}E. \tag{19}$$

In the FSU scheme, on the other hand, the stationarity requires

$$\mathcal{T}\,(|\hat{U}\rangle\otimes|U\rangle)=|U\rangle\otimes|\hat{U}\rangle, \tag{20}$$

which implies that the defect $|\hat{U}\rangle$ is now transferred in the forward direction. Equation (20) leads to a quadratic algebra, which is identical to Eqs. (19) upon the replacements $A\leftrightarrow\hat{A}$, $D\leftrightarrow\hat{D}$, $E\leftrightarrow\hat{E}$.

D. Mapping onto the RSU algebra

In order to calculate physical quantities using the MPS, it is in principle necessary to find a representation of the matrices that satisfies the relevant quadratic algebras for the BSU and FSU schemes. In some cases, however, this job can be avoided and the quadratic algebra itself is sufficient for calculations [8,11,13,17,18,28] . Also recent studies on ASEP [21,22] and multispecies ASEP [28–30] have demonstrated that the algebra for a stochastic process in an OSU scheme can be mapped onto the algebra for a related stochastic process in the RSU scheme, for which an explicit representation of the matrices are known. Here we show that this mapping holds for the two-way traffic model as well. We first assume

$$\hat{A}=A+a, \quad \hat{D}=D+d, \quad \hat{E}=E+e, \tag{21}$$

where a, d, and e are real numbers. In the BSU scheme, it can be verified that the choice

$$a=0, \quad d=-\frac{\eta}{\beta}, \quad e=\frac{\eta}{\beta-\eta} \tag{22}$$

map the algebra (19) to

$$DE=D+E, \quad \tilde{\alpha}AE=A, \quad \tilde{\beta}DA=A, \tag{23}$$

where

$$\tilde{\alpha}=\frac{\alpha(\beta-\eta)}{\beta-\alpha\eta}, \quad \tilde{\beta}=\beta. \tag{24}$$

Note that algebra (23) is identical to the RSU algebra (6) except for the renormalization of α to $\tilde{\alpha}$.

In the FSU scheme, one may use the mapping to the BSU algebra by simply interchanging the roles of A, D, and E with $\hat{A}$, $\hat{D}$, and $\hat{E}$ [see Eqs. (18) and (20)]. However, it turns out that it is more convenient for the subsequent analysis to use a direct mapping to the RSU scheme. We again assume Eq. (21), and choose

$$a=0, \quad d=\frac{\eta}{\beta-\eta}, \quad e=-\frac{\eta}{\beta}. \tag{25}$$

Then the FSU algebra is mapped to Eqs. (23) with

$$\tilde{\alpha}=\alpha, \quad \tilde{\beta}=\frac{\beta-\eta}{1-\eta}. \tag{26}$$

Note that now β is renormalized to $\tilde{\beta}$.

III. AVERAGE VELOCITIES

In this section we consider the special case where there is only a single truck in the system, and evaluate the average velocities of cars and the truck.

A. BSU scheme

The movement of cars from a site, k for example, to its neighboring site $k+1$ is achieved by $T_{k,k+1}$. Then the average current of cars $\langle J_k^{\mathrm{car}}\rangle$ between the two sites can be written as follows:

$$\langle J_k^{\mathrm{car}}\rangle=\frac{1}{Z(N,M)}\left\{\eta\sum_{\mathrm{conf}}^{N,M}\mathrm{Tr}(\underbrace{\cdots}_{k-1\ \mathrm{sites}}D\hat{A}\cdots)+\frac{\eta}{\beta}\sum_{\mathrm{conf}}^{N,M}\mathrm{Tr}(\underbrace{\cdots}_{k-1\ \mathrm{sites}}D\hat{E}\cdots)\right\} \tag{27}$$

where the specifications N and M in the summations represent that the matrix products $(\cdots D\hat{A}\cdots)$ and $(\cdots D\hat{E}\cdots)$ contain N matrices and M of them denote cars. Below, these specifications will be omitted in usual situations, that is, when they are N and M. In Eq. (27), it has been taken into account that before the action of $T_{k,k+1}$, the steady state is acted upon by $T_{k+1,k+2}\cdots T_{N-1,N}$, which shifts the hat from the site N to $k+1$. More rigorous derivation is given in the Appendix. In a similar way, the average current of a truck $\langle J_k^{\mathrm{truck}}\rangle$ between the site k and $k+1$ reads

$$\langle J_k^{\mathrm{truck}}\rangle=\frac{1}{Z(N,M)}\left\{\eta\gamma\sum_{\mathrm{conf}}\mathrm{Tr}(\underbrace{\cdots}_{k-1}A\hat{E}\cdots)+\frac{\eta}{\beta}\sum_{\mathrm{conf}}\mathrm{Tr}(\underbrace{\cdots}_{k-1}D\hat{E}\cdots)\right\}. \tag{28}$$

It is instructive to briefly sketch the derivation of Eq. (27). Using the cyclic invariance and the fact that in the first term of Eq. (27), the single truck can locate anywhere on the remaining $N-2$ sites, we find

$$\langle J_k^{\rm car}\rangle = \frac{1}{Z(N,M)}\left\{\eta\sum_{i=1}^{N-2}\sum_{\rm conf}{\rm Tr}(\underbrace{\cdots}_{i-1}D\hat{A}\cdots E)+\frac{\eta}{\beta}\sum_{\rm conf}{\rm Tr}(\cdots D\hat{E})\right\} . \tag{29}$$

Note that the site index k has disappeared from the expression and thus the current is independent of k, as it should be from the conservation of the number of cars. Recalling that $\hat{A}=A$ and $\tilde{\beta}DA=A$, the first term becomes

$$\frac{\eta}{\tilde{\beta}}\sum_{i=1}^{N-2}\sum_{\rm conf}^{N-1,M-1}{\rm Tr}(\underbrace{\cdots}_{i-1}A\cdots E) .$$

In this double summation, each configuration $(\cdots E)$ is counted $[(N-1)-(M-1)-1]$ times, and thus it can be simplified further to

$$\frac{\eta}{\tilde{\beta}}(N-M-1)Y(N-1,M-1),$$

where

$$Y(P,Q)=\sum_{\rm conf}^{P,Q}{\rm Tr}(\cdots E). \tag{30}$$

Also recalling that $\hat{E}=E+e$, the second term in Eq. (29) can be split into two pieces:

$$\frac{\eta}{\beta}Y_D(N,M)+e\frac{\eta}{\beta}\sum_{\rm conf}^{N-1,M}{\rm Tr}(\underbrace{\cdots D}_{0\ \rm truck}) \tag{31}$$

where

$$Y_D(P,Q)=\sum_{\rm conf}^{P,Q}{\rm Tr}(\cdots DE). \tag{32}$$

Using $\tilde{\beta}DA=A$ and the cyclic invariance of the trace, the second piece in Eq. (31) can be simplified to

$$e\frac{\eta}{\beta}\frac{1}{\tilde{\beta}^M}C_{N-2}^{M-1}{\rm Tr}(A^{N-M-1}),$$

where the binomial coefficient C_{N-2}^{M-1} comes from counting the number of configurations that satisfy the given specifications. Later it turns out that the factor ${\rm Tr}(A^{N-M-1})$ cancels out for all expressions for physical quantities including $\langle J_k^{\rm car}\rangle$ and $\langle J_k^{\rm truck}\rangle$. Below we thus set ${\rm Tr}(A^{N-M-1})=1$ for convenience. We also mention that due to the cancellation, all calculations in this paper can be performed without resorting to an explicit representation of the matrices.

To evaluate the denominator of Eq. (27), we proceed with definition (17):

$$Z(N,M)=\sum_{\rm conf}{\rm Tr}(\cdots\hat{E})+\sum_{\rm conf}{\rm Tr}(\cdots\hat{D})+\sum_{\rm conf}{\rm Tr}(\cdots\hat{A}). \tag{33}$$

In a similar evaluation procedure as above, the first term on the right-hand side yields

$$\sum_{\rm conf}{\rm Tr}(\cdots\hat{E})=Y(N,M)+\frac{e}{\tilde{\beta}^M}C_{N-1}^M. \tag{34}$$

Also the sum of the second and third terms on the right-hand side becomes

$$\sum_{i=1}^{N-1}\sum_{\rm conf}\left\{{\rm Tr}(\underbrace{\cdots}_{i-1}\hat{D}\cdots E)+{\rm Tr}(\underbrace{\cdots}_{i-1}\hat{A}\cdots E)\right\} , \tag{35}$$

which results in

$$(N-1)Y(N,M)+d(N-1)Y(N-1,M-1). \tag{36}$$

Combining all calculations given above, one obtains the exact expression for $\langle J_k^{\rm car}\rangle$ for arbitrary N and M:

$$\langle J_k^{\rm car}\rangle=\frac{1}{Z(N,M)}\left\{\frac{\eta}{\tilde{\beta}}(N-M-1)Y(N-1,M-1)+\frac{\eta}{\beta}Y_D(N,M)+e\frac{\eta}{\beta}\frac{1}{\tilde{\beta}^M}C_{N-2}^{M-1}\right\}. \tag{37}$$

For a complete evaluation of Eq. (27), one needs $Y(N,M)$ and $Y_D(N,M)$. Exactly the same quantities are calculated in the RSU scheme [11]. Taking into account the renormalization of α and β, one immediately obtains

$$Y_D(N,M)=\frac{1}{\tilde{\alpha}\tilde{\beta}^M}\left[\frac{\tilde{\alpha}}{1-\tilde{\beta}}C_{N-2}^{M-1}+\frac{\tilde{\alpha}+\tilde{\beta}-1}{\tilde{\beta}-1}I(N,M)\right], \tag{38}$$

$$Y(N,M)=Y_D(N,M)+\frac{1}{\tilde{\alpha}\tilde{\beta}^M}C_{N-2}^M,$$

$$I(N,M)=\sum_{k=1}^{M}\tilde{\beta}^kC_{N-2-k}^{M-k}. \tag{39}$$

To study the implications of Eq. (37), we consider the thermodynamic limit $N,M\rightarrow\infty$ while $n=M/N$ is kept fixed. As demonstrated in Ref. [11], the thermodynamic limit is governed by $I(N,M)$. Using the method of steepest descent, it can be verified that, for $n\tilde{\beta}>1$,

$$I(N,M)\to\frac{\tilde{\beta}^{N-1}}{(\tilde{\beta}-1)^{N-M-1}},$$

which is exponentially larger than all other terms and hence dominates the thermodynamic behavior of all quantities. For $n\tilde{\beta}<1$, on the other hand,

$$I(N,M)\to\frac{n\tilde{\beta}(1-n)^2}{1-n\tilde{\beta}}C_N^M,$$

which is of the same order as other terms. A straightforward calculation then shows that $\langle v_{\rm car}\rangle=(1/n)\langle J_k^{\rm car}\rangle$ is given by

$$\langle v_{\rm car}\rangle=\begin{cases}\dfrac{\eta}{1-\eta n}(1-n) & \text{if}\quad n\tilde{\beta}\leqslant 1\\[2mm] \dfrac{\eta}{\beta-\eta}\,\dfrac{1-n}{n} & \text{if}\quad n\tilde{\beta}\geqslant 1.\end{cases}\tag{40}$$

The average current of the single truck [Eq. (28)] can be evaluated in a similar manner:

$$\langle J_k^{\rm truck}\rangle=\frac{1}{Z(N,M)}\left\{\eta\gamma\left[Y(N,M)-Y_D(N,M)+\frac{e}{\tilde{\beta}^M}C_{N-2}^M\right]\right.\\ \left.+\frac{\eta}{\beta}\left[Y_D(N,M)+\frac{e}{\tilde{\beta}^M}C_{N-2}^{M-1}\right]\right\}.\tag{41}$$

In the thermodynamic limit, the average velocity $\langle v_{\rm truck}\rangle=N\langle J_k^{\rm truck}\rangle$ becomes

$$\langle v_{\rm truck}\rangle=\begin{cases}\dfrac{\eta}{\beta}\,\dfrac{\{\alpha(1-n)+e\tilde{\alpha}[n+\alpha(1-n)]\}(1-n\tilde{\beta})+n(\tilde{\alpha}+\tilde{\beta}-n\tilde{\beta})}{(1-\eta n)[(1-n)(1-n\tilde{\beta})+n(\tilde{\alpha}+\tilde{\beta}-n\tilde{\beta})]} & \text{if}\quad n\tilde{\beta}\leqslant 1\\[2mm] \dfrac{\eta}{\beta-\eta} & \text{if}\; n\tilde{\beta}\geqslant 1.\end{cases}\tag{42}$$

Figures 1 and 2 show the behaviors of $\langle v_{\rm car}\rangle$ and $\langle v_{\rm truck}\rangle$ as a function of $1-(1/\beta)$, the narrowness of the road, while the values of $\eta,\gamma(=\alpha/\beta)$ and n are fixed. The behavior of $\langle v_{\rm car}\rangle$ changes clearly at the transition point $1-(1/\beta_c)=1-n$. Below the transition point, $\langle v_{\rm car}\rangle$ is constant, implying that the interaction with the truck causes negligible changes on the flow of cars. Above the transition point, on the other hand, $\langle v_{\rm car}\rangle$ begins to drop suddenly generating a cusp at the transition point. Thus a local dynamic impurity, truck, results in global effects. Here the decrease of $\langle v_{\rm car}\rangle$ follows a hyperbolic curve. $\langle v_{\rm truck}\rangle$ also changes its behavior at the transition point. Below the transition point, one finds a rather smooth decrease in $\langle v_{\rm truck}\rangle$ and above the transition point, one again observes a hyperbolic decrease.

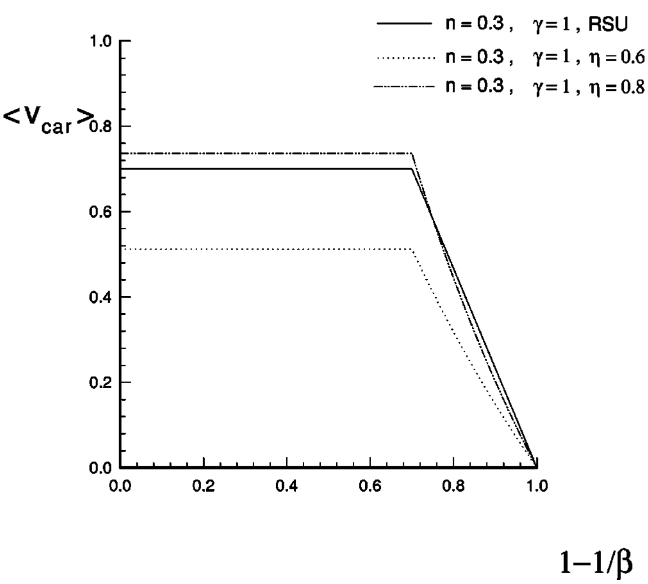

FIG. 1. Average velocity of cars in the BSU and RSU schemes for $n=0.3$ and different values of η. The velocity is measured in units of number of sites per time step (BSU scheme), and number of sites per unit time (RSU scheme), respectively. The road narrowness $1-1/\beta$ in the horizontal axis is dimensionless.

B. FSU scheme

In the FSU scheme, the average current of cars $\langle J_k^{\rm car}\rangle$ between the site k and $k+1$ reads

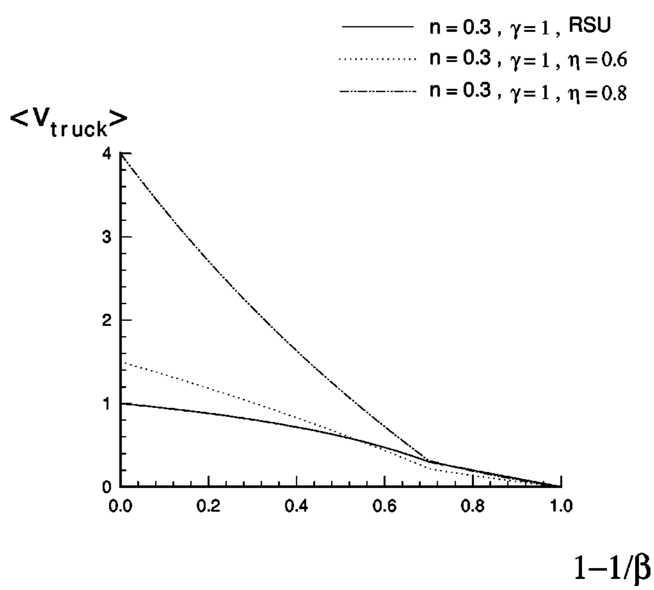

FIG. 2. Average velocity of the truck in the BSU and RSU schemes for $n=0.3$ and different values of η. Units for the horizontal and the vertical axes are the same as in Fig. 1.

$$\langle J_k^{\mathrm{car}}\rangle = \frac{1}{Z(N,M)}\left\{\eta\sum_{\mathrm{conf}}\mathrm{Tr}(\underbrace{\cdots}_{k-1}\hat{D}A\cdots)+\frac{\eta}{\beta}\sum_{\mathrm{conf}}\mathrm{Tr}(\underbrace{\cdots}_{k-1}\hat{D}E\cdots)\right\}. \tag{43}$$

In comparison with Eq. (27) for the BSU scheme, it should be noted that the hats now appear at the site k instead of $k+1$ since the action of $T_{k+1,k}$ occurs after the steady state is acted upon by $T_{k,k-1}\cdots T_{2,1}T_{1,N}$. This expression can be derived in a more rigorous way following a similar procedure to that sketched in the Appendix. Using the quadratic algebra of the matrices, this expression can be reduced to

$$\langle J_k^{\mathrm{car}}\rangle = \frac{1}{Z(N,M)}\left\{\eta\left(\frac{1}{\tilde{\beta}}+d\right)(N-M-1)Y(N-1,M-1)+\frac{\eta}{\beta}[Y_D(N,M)+dY(N-1,M-1)]\right\}. \tag{44}$$

Note that the k dependence has disappeared. Here it should be understood that $\tilde{\alpha}$, $\tilde{\beta}$, d, and e now have the values given in Eqs. (25) and (26). The same understanding is also needed for $Y(N,M)$, $Y_D(N,M)$, and $Z(N,M)$ whose explicit expressions are given in Sec. III A.

Similar shifts of the hats occur for the average current of the truck as well, and one finds

$$\langle J_k^{\mathrm{truck}}\rangle = \frac{1}{Z(N,M)}\left\{\eta\gamma\sum_{\mathrm{conf}}\mathrm{Tr}(\underbrace{\cdots}_{k-1}\hat{A}E\cdots)+\frac{\eta}{\beta}\sum_{\mathrm{conf}}\mathrm{Tr}(\underbrace{\cdots}_{k-1}\hat{D}E\cdots)\right\}, \tag{45}$$

which is equal to

$$\langle J_k^{\mathrm{truck}}\rangle = \frac{1}{Z(N,M)}\left\{\eta\gamma[Y(N,M)-Y_D(N,M)]+\frac{\eta}{\beta}[Y_D(N,M)+dY(N-1,M-1)]\right\}. \tag{46}$$

Note again that the k dependence has disappeared.

Using the relations $\langle v_{\mathrm{car}}\rangle=(1/n)\langle J_k^{\mathrm{car}}\rangle$ and $\langle v_{\mathrm{truck}}\rangle=N\langle J_k^{\mathrm{truck}}\rangle$, and taking the thermodynamic limit, one reaches the following results

$$\langle v_{\mathrm{car}}\rangle=\begin{cases}\dfrac{\eta(1-n)}{1-\eta(1-n)} & \text{if } n\tilde{\beta}\leqslant 1\\[2ex] \dfrac{\eta}{\beta}\dfrac{(1-n)}{n} & \text{if } n\tilde{\beta}\geqslant 1\end{cases} \tag{47}$$

$$\langle v_{\mathrm{truck}}\rangle=\begin{cases}\dfrac{\eta}{\beta}\dfrac{(\tilde{\alpha}+dn\tilde{\beta})(1-n)(1-n\tilde{\beta})+(1+dn\tilde{\beta})n(\tilde{\alpha}+\tilde{\beta}-n\tilde{\beta})}{(1+dn\tilde{\beta})[(1-n)(1-n\tilde{\beta})+n(\tilde{\alpha}+\tilde{\beta}-n\tilde{\beta})]} & \text{if } n\tilde{\beta}\leqslant 1\\[2ex] \dfrac{\eta}{\beta} \quad \text{if } n\tilde{\beta}\geqslant 1. \end{cases} \tag{48}$$

Figures 3 and 4 show $\langle v_{\mathrm{car}}\rangle$ and $\langle v_{\mathrm{truck}}\rangle$ as a function of the narrowness $1-(1/\beta)$, while $\eta,\gamma(=\alpha/\beta)$, and n are fixed. One again finds that the average velocities have cusps at the transition point. Above the critical narrowness, both $\langle v_{\mathrm{car}}\rangle$ and $\langle v_{\mathrm{truck}}\rangle$ decrease linearly with respect to the narrowness, which is in contrast to the hyperbolic decreases in the BSU scheme. Below the critical narrowness, $\langle v_{\mathrm{car}}\rangle$ is constant, similar to the result in the BSU scheme. But its value is different from the corresponding one in the BSU scheme.

C. Comparison with the RSU results

In the thermodynamic limit, $\langle v_{\mathrm{car}}\rangle_{\mathrm{RSU}}$ and $\langle v_{\mathrm{truck}}\rangle_{\mathrm{RSU}}$ become [11]

$$\langle v_{\mathrm{car}}\rangle_{\mathrm{RSU}}=\begin{cases}1-n & \text{if } n\beta\leqslant 1\\[1ex] \dfrac{1}{\beta}\dfrac{(1-n)}{n} & \text{if } n\beta\geqslant 1\end{cases} \tag{49}$$

$$\langle v_{\mathrm{truck}} \rangle_{\mathrm{RSU}} = \begin{cases} \dfrac{1}{\beta} \dfrac{\alpha(1-n)(1-n\beta)+n(\alpha+\beta-n\beta)}{(1-n)(1-n\beta)+n(\alpha+\beta-n\beta)} & \text{if} \quad n\beta \leqslant 1 \\ \dfrac{1}{\beta} \quad \text{if} \quad n\beta \geqslant 1. \end{cases} \tag{50}$$

We now compare the results. In all updating schemes considered, a phase transition occurs at a critical value of the narrowness. In the RSU and BSU schemes, the critical value is $1-n$ while in the FSU scheme, it is $[(1-\eta)(1-n)]/[1-\eta(1-n)]$. Note that the transition point can be significantly lower in the FSU scheme when $\eta \approx 1$. Below the critical narrowness, $\langle v_{\mathrm{car}} \rangle$ is independent of β in all three schemes (with different values in each scheme) and above it, $\langle v_{\mathrm{car}} \rangle$ decreases linearly in the RSU and FSU schemes and hyperbolically in the BSU scheme. $\langle v_{\mathrm{truck}} \rangle$, on the other hand, is not constant even below the critical narrowness in all three schemes, and varies quite smoothly with respect to the narrowness. Above the critical narrowness, $\langle v_{\mathrm{truck}} \rangle$ decreases linearly in the RSU and FSU schemes and hyperbolically in the BSU scheme.

In order to illustrate the origin of these differences, it is instructive to consider the behavior of $\langle v_{\mathrm{truck}} \rangle$ in the vanishing car density limit $n \to 0$. In this limit, $\langle v_{\mathrm{truck}} \rangle$ in the RSU and FSU schemes approaches γ and $\eta\gamma$, respectively, while it approaches $\eta\gamma/(1-\eta\gamma)$ in the BSU scheme. This difference can be explained in the following way. In the absence of any car, the single truck in the RSU scheme either hops with the probability γdt or stays at the present site with the probability $1-\gamma dt$ during a time interval dt. Hence its velocity is equal to its hopping rate γ. The situation is similar in the FSU scheme. Since the truck moves in the opposite direction of the update, it either hops one site ahead with the probability $\eta\gamma$ or stays with the probability $(1-\eta\gamma)$. Therefore, the average velocity reads

$$\langle v_{\mathrm{truck}} \rangle = \frac{\text{average distance}}{\text{number of time steps}}$$
$$= \frac{0 \times (1-\eta\gamma) + 1 \times (\eta\gamma)}{1} = \eta\gamma.$$

The situation changes drastically in the BSU scheme. In this case, the direction of the update and the direction of the truck movement are identical and accordingly the truck can be transferred by large distances in a single time step. Recalling that each hopping occurs with the probabilities $\eta\gamma$ and the probability of stopping is $1-\eta\gamma$, $\langle v_{\mathrm{truck}} \rangle$ can be cast into the form

$$0 \times (1-\eta\gamma) + \eta\gamma(1-\eta\gamma) + 2(\eta\gamma)^2(1-\eta\gamma) + 3(\eta\gamma)^3(1-\eta\gamma) + \cdots,$$

which is simplified to $\eta\gamma/(1-\eta\gamma)$.

We next examine the fundamental diagrams (the relation between the car current and the car density) in three updating schemes. The average current of cars is simply related to the average velocity via $\langle J_{\mathrm{car}} \rangle = n \langle v_{\mathrm{car}} \rangle$. Using the thermodynamic behaviors of $\langle v_{\mathrm{car}} \rangle$ [Eqs. (40), (47), and (49)], one can determine the thermodynamic behaviors of the current as a function of the density. Figure 5 shows these behaviors for some constant values of β and η in different updating schemes. In all three types of update, the single truck does

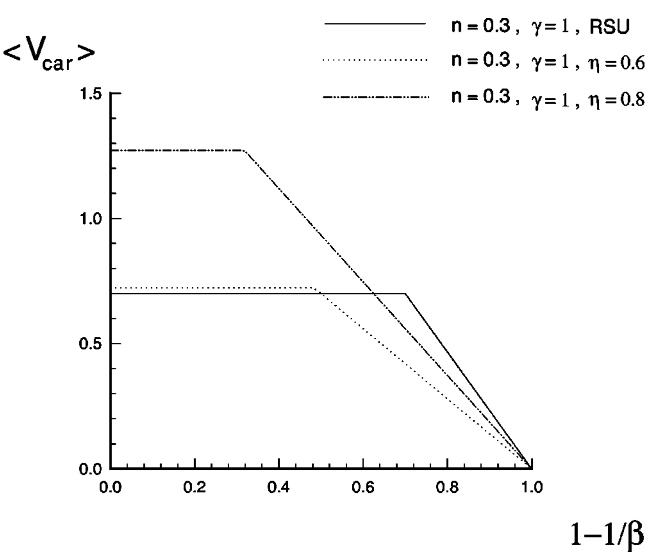

FIG. 3. Average velocity of cars in the FSU and RSU schemes for $n=0.3$ and different values of η. The velocity is measured in units of number of sites per time step (FSU scheme), and number of sites per unit time (RSU scheme), respectively. The road narrowness $1-1/\beta$ in the horizontal axis is dimensionless.

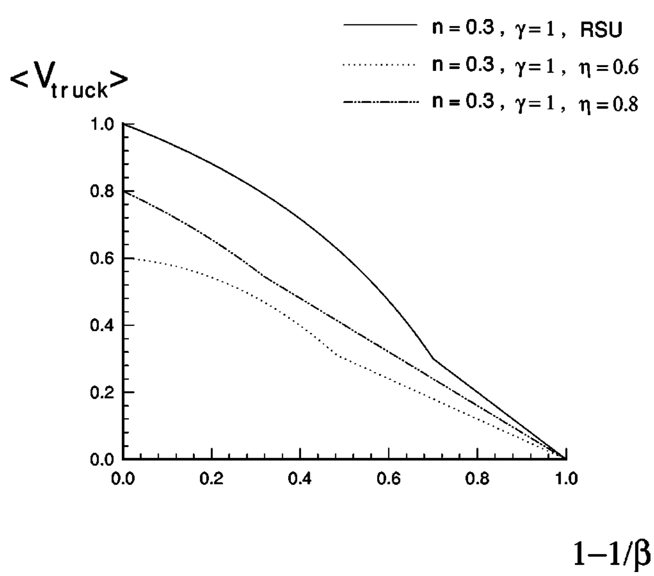

FIG. 4. Average velocity of the truck in the FSU and RSU schemes for $n=0.3$ and different values of η. Units for the horizontal and the vertical axes are the same as in Fig. 3.

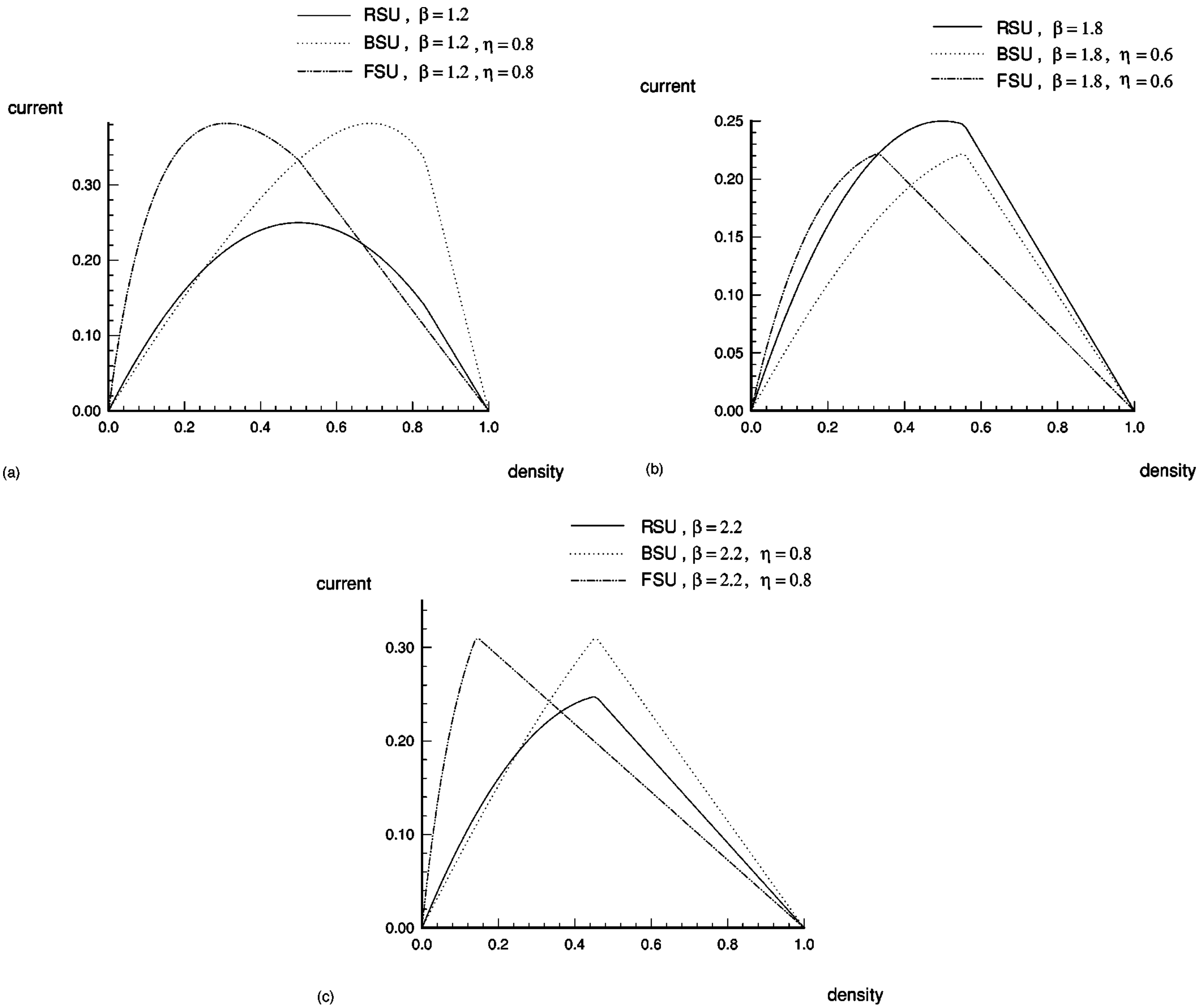

FIG. 5. Fundamental diagrams in three updating schemes for different values of β and η (values are given in the figures). The density in the horizontal axis is measured in units of number of cars per site. The current in the vertical axis is measured in units of number of cars per time step (BSU and FSU schemes), and number of cars per unit time (RSU scheme), respectively.

not affect the current-density diagram before the transition point, whereas it affects the system after the transition point in a nontrivial manner: linear decrease of the current with increasing density.

IV. DENSITY PROFILE

A. Density profile in the RSU scheme

Here we first summarize the results in Ref. [11]. In the RSU scheme, it can be assumed without loss of generality that the single truck is at a particular site, for example at the site N of the chain due to the cyclic invariance of the problem. Thus the probability to find a car at the distance x from the truck can be written via MPS as follows:

$$\langle n(x)\rangle_{\mathrm{RSU}} = \frac{\sum_{\mathrm{conf}} \mathrm{Tr}(\cdots D\overbrace{\cdots}^{x} E)}{\sum_{\mathrm{conf}} \mathrm{Tr}(\cdots\cdots E)}\ . \tag{51}$$

Using the matrix algebra (6), one finds

$$\langle n(x)\rangle_{\mathrm{RSU}} = [Y_{\mathrm{RSU}}(N,M)\alpha\beta^{M}]^{-1}\Bigg\{ C_{N-3}^{M-1} - \frac{\alpha}{\beta-1} C_{N-3}^{M-2} + \frac{\alpha+\beta-1}{\beta-1}[I_{\mathrm{RSU}}(N-1,M-1) + \beta^{x-1}(\beta-1) I_{\mathrm{RSU}}(N-x,M-x)\theta(M\geqslant x)]\Bigg\}, \tag{52}$$

where $Y_{\mathrm{RSU}}(N,M)$ and $I_{\mathrm{RSU}}(N,M)$ can be obtained from $Y(N,M)$ and $I(N,M)$ [Eqs. (38) and (39)] by replacing $\tilde{\alpha}$ and $\tilde{\beta}$ with α and β, respectively, and the factor $\theta(y\geqslant x)$ is 1 if $y\geqslant x$ and 0 otherwise. In the high density phase $n\beta \geqslant 1$, the single truck affects the system globally. The density profile is

$$\langle n(x)\rangle_{\mathrm{RSU}}=\begin{cases}1 & \text{for } \dfrac{x}{N}\leqslant l_{\mathrm{RSU}}\equiv\dfrac{n\beta-1}{\beta-1}\\ \dfrac{1}{\beta} & \text{otherwise.}\end{cases} \tag{53}$$

Note that the system consists of two regions: a traffic jam region in front of the truck and a free flow region behind it. In the low density phase $n\beta\leqslant 1$, on the other hand, the presence of the truck has only local effects. In the thermodynamic limit, the car density becomes

$$\langle n(x)\rangle_{\mathrm{RSU}}=n\left\{1+\frac{(\alpha+\beta-1)(1-n)}{1-n+\alpha n}(n\beta)^x\right\}, \tag{54}$$

which shows that the disturbance by the truck decays exponentially with a characteristic length scale

$$\xi_{\mathrm{RSU}}=|\ln(n\beta)|^{-1}. \tag{55}$$

B. Density profile in the ordered sequential updating schemes

As stated in Sec. II, the ordered sequential updates break the cyclic invariance. In what follows we show that due to the absence of the cyclic invariance, the probability to find a car at x sites in front of the truck varies depending on the truck location. Two updating schemes, BSU and FSU, will be considered simultaneously since same expressions apply to both schemes.

We first consider the case where the truck is at the site N. The conditional probability to find a car at x sites in front of the truck reads

$$\mathrm{Prob}(N-1-x=\mathrm{car}|N=\mathrm{truck})=\frac{\sum_{\mathrm{conf}}\mathrm{Tr}(\cdots D\overbrace{\cdots}^{x}\hat{E})}{\sum_{\mathrm{conf}}\mathrm{Tr}(\cdots\hat{E})}. \tag{56}$$

Here it is helpful to define a quantity $K(x,N,M)$ by

$$K(x,N,M)=\frac{\sum_{\mathrm{conf}}\mathrm{Tr}(\cdots D\overbrace{\cdots}^{x}E)}{\sum_{\mathrm{conf}}\mathrm{Tr}(\cdots\cdots E)},$$

which is exactly the same as Eq. (51). All properties of $K(x,N,M)$ can be thus obtained from Eq. (52) by replacing α and β with $\tilde{\alpha}$ and $\tilde{\beta}$, respectively. In terms of $K(x,N,M)$, the conditional probability (56) becomes

$$\frac{Y(N,M)K(x,N,M)+\dfrac{e}{\tilde{\beta}^M}C_{N-2}^{M-1}}{Y(N,M)+\dfrac{e}{\tilde{\beta}^M}C_{N-1}^{M}}. \tag{57}$$

The thermodynamic limit can be investigated in a simple way. In the high density phase $n\tilde{\beta}\geqslant 1$, the second terms both in the numerator and the denominator in Eq. (57) are negligible compared to the first terms and Eq. (57) reduces to $K(x,N,M)$. Thus one finds

$$\mathrm{Prob}(N-x-1=\mathrm{car}|N=\mathrm{truck})=\begin{cases}1 & \text{for } \dfrac{x}{N}\leqslant l\equiv\dfrac{n\tilde{\beta}-1}{\tilde{\beta}-1}\\ \dfrac{1}{\tilde{\beta}} & \text{otherwise}\end{cases} \tag{58}$$

which is essentially identical to result (53) in the RSU scheme except for the replacement of β by $\tilde{\beta}$. In the low density phase $n\tilde{\beta}\leqslant 1$, on the other hand, the second terms are comparable with the first terms, and one obtains

$$\mathrm{Prob}(N-x-1=\mathrm{car}|N=\mathrm{truck})=n\left\{1+\frac{(\tilde{\alpha}+\tilde{\beta}-1)(1-n)}{(1-n+\tilde{\alpha}n)+e(1-n\tilde{\beta})\tilde{\alpha}}(n\tilde{\beta})^x\right\}, \tag{59}$$

which shows the exponential decay of the disturbance with a length scale $\xi=|\ln(n\tilde{\beta})|^{-1}$. The coefficient of the exponential decaying term is different from the result in the RSU scheme.

Secondly we consider the case where the truck is at the site $N-k$ $(1\leqslant k\leqslant N-x-2)$. The conditional probability becomes

$$\mathrm{Prob}(N-x-k-1=\mathrm{car}|N-k=\mathrm{truck})=\frac{\sum_{\mathrm{conf}}\mathrm{Tr}(\cdots D\overbrace{\cdots}^{x}E\overbrace{\cdots}^{k-1}\hat{X}_N)}{\sum_{\mathrm{conf}}\mathrm{Tr}(\cdots E\underbrace{\cdots}_{k-1}\hat{X}_N)}. \tag{60}$$

The evaluation of Eq. (60) using the matrix algebra leads to

$$\frac{Y(N,M)K(x,N,M)+dY(N-1,M-1)K(x,N-1,M-1)}{Y(N,M)+dY(N-1,M-1)}. \tag{61}$$

Note that the conditional probability is independent of k.

In the thermodynamic limit, Eq. (61) can be greatly simplified to $K(x,N,M)$ since $K(x,N,M)$ and $K(x,N-1,M-1)$ in the numerator become identical in this limit [see Eqs. (53) and (54)]. Thus the conditional probability $\text{Prob}(N-x-k-1=\text{car}|N-k=\text{truck})$ is the same as the corresponding results (53) and (54) in the high and the low density phases except for the renormalization of α and β to $\tilde{\alpha}$ and $\tilde{\beta}$.

In the third case, the truck is located at the site $x+1$ and the car at the site N. The conditional probability becomes

$$\text{Prob}(N=\text{car}|x+1=\text{truck})=\frac{\sum_{\text{conf}}\text{Tr}(\overbrace{\cdots}^{x}E\cdots\hat{D})}{\sum_{\text{conf}}\text{Tr}(\underbrace{\cdots}_{x}E\cdots\hat{X}_N)}, \tag{62}$$

which is found to be

$$\frac{Y(N,M)K(x,N,M)+dY(N-1,M-1)}{Y(N,M)+dY(N-1,M-1)}. \tag{63}$$

Note that the second terms both in the numerator and denominator are comparable to the first terms in the thermodynamic limit. As a result, the conditional probability becomes noticeably different from $K(x,N,M)$. In the high density phase, one finds

$$\text{Prob}(N=\text{car}|x+1=\text{truck})=\begin{cases}1 & \text{for } \dfrac{x}{N}\leqslant l=\dfrac{n\tilde{\beta}-1}{\tilde{\beta}-1}\\[2ex] \dfrac{1}{\tilde{\beta}}\dfrac{1+d\tilde{\beta}}{1+d} & \text{otherwise,}\end{cases} \tag{64}$$

and, in the low density phase,

$$\text{Prob}(N=\text{car}|x+1=\text{truck})=\frac{n(1+d\tilde{\beta})}{1+dn\tilde{\beta}}\left\{1+\frac{(\tilde{\alpha}+\tilde{\beta}-1)(1-n)}{(1+d\tilde{\beta})(1-n+\tilde{\alpha}n)}(n\tilde{\beta})^x\right\}. \tag{65}$$

In contrast to the previous cases, the renormalization of α and β are not sufficient to account for deviations from the RSU results even in the high density phase.

Finally we consider $\text{Prob}(N-l=\text{car}|x-l+1=\text{truck})$ $(1\leqslant l\leqslant x)$, which takes the following form:

$$\frac{Y(N,M)K(x,N,M)+dY(N-1,M-1)K(x-1,N-1,M-1)}{Y(N,M)+dY(N-1,M-1)}. \tag{66}$$

Note that it is independent of l. The thermodynamic limit can be examined in a similar way. In the high density phase, $K(x,N,M)$ and $K(x-1,N-1,M-1)$ are essentially identical and thus Eq. (66) reduces to Eq. (53) for the RSU scheme except for the trivial replacement of α and β with $\tilde{\alpha}$ and $\tilde{\beta}$. In the low density phase, however, the difference between $K(x,N,M)$ and $K(x-1,N-1,M-1)$ is not negligible, and careful treatment is required to take care of the difference. This way, one obtains

$$\text{Prob}(N-l=\text{car}|x-l+1=\text{truck})=n\left\{1+\frac{1+d}{1+dn\tilde{\beta}}\frac{(\tilde{\alpha}+\tilde{\beta}-1)(1-n)}{1-n+\tilde{\alpha}n}(n\tilde{\beta})^x\right\}. \tag{67}$$

We next discuss the origin of the absence of the cyclic invariance, which results in the four different cases. As stated above, the system loses the cyclic invariance due to the choice of a particular site, the site N, as a starting point of the update.

Thus by choosing the starting point in an even way, the cyclic invariance can be restored. One can, for example, define a cyclically invariant average $\langle\langle\cdots\rangle\rangle$ of an operator $\hat{O}$ in the following way:

$$\langle\langle\hat{O}\rangle\rangle=\frac{1}{N}\sum_{k=1}^{N}\langle s|\hat{O}|P_{s,k}\rangle, \tag{68}$$

where $\langle s|=\Sigma_{\{\tau\}}\langle\{\tau\}|$ and $|P_{s,k}\rangle_k$ is the stationary state of the transfer matrix

$$T_{\leftarrow,k} \equiv T_{k,k+1}T_{k+1,k+2}\cdots T_{N,1}T_{1,2}\cdots T_{k-1,k}$$

in the BSU scheme or

$$T_{\rightarrow,k} \equiv T_{k,k-1}T_{k-1,k-2}\cdots T_{1,N}T_{N,N-1}\cdots T_{k+1,k}$$

in the FSU scheme with the site k as a starting point of the update. (This problem due to the absence of the cyclic invariance does not affect the calculation of the currents since the currents should be independent of the sites, where their values are evaluated due to the conservation of the particle number.) Note that the site N now loses its special meaning and the cyclic invariance becomes evident in Eq. (68). It is also worth mentioning that definition (68) is equivalent to taking an expectation value at each substep of an update, that is, after the action of $T_{l,l+1}$ or $T_{l+1,l}$ instead of a single whole step of the update $T_{\leftarrow}$ or $T_{\rightarrow}$, and taking the average over these expectation values.

Using definition (68), we calculate the density $\langle\langle n(x)\rangle\rangle = \langle\langle \delta_{\tau_{N-l},2}\delta_{\tau_{N-l-x-1},1}\rangle\rangle / \langle\langle \delta_{\tau_{N-l},2}\rangle\rangle$ $(0 \leqslant l \leqslant N-1)$, where the sites $0,-1,-2,\ldots$ should be identified with the sites $N,N-1,N-2,\ldots$, respectively. The density $\langle\langle n(x)\rangle\rangle$ is, by definition, independent of the truck location $N-l$ and thus one may choose $l=0$. The denominator $\langle\langle \delta_{\tau_N,2}\rangle\rangle$ is $1/N$ due to the cyclic invariance, and the numerator $\langle\langle \delta_{\tau_N,2}\delta_{\tau_{N-x-1},1}\rangle\rangle$ becomes

$$\frac{1}{NZ(N,M)}\Bigg\{\sum_{\rm conf}\mathrm{Tr}(\cdots D\underbrace{\cdots}_{x}\hat{E}) + \sum_{k=0}^{N-x-3}\sum_{\rm conf}\mathrm{Tr}(\underbrace{\cdots}_{k}\hat{X}_{k+1}\cdots D\underbrace{\cdots}_{x}E)$$
$$+\sum_{\rm conf}\mathrm{Tr}(\cdots\hat{D}\underbrace{\cdots}_{x}E) + \sum_{l=0}^{x-1}\sum_{\rm conf}\mathrm{Tr}(\cdots D\underbrace{\cdots}_{l}\hat{X}_{N-x+l}\underbrace{\cdots}_{x-l-1}E)\Bigg\}. \tag{69}$$

The cyclically invariant density then becomes

$$\langle\langle n(x)\rangle\rangle = \frac{1}{Z(N,M)}\Bigg[NY(N,M)K(x,N,M) + \frac{e}{\tilde{\beta}^M}C_{N-2}^{M-1} + dY(N-1,M-1)$$
$$+d(N-x-2)Y(N-1,M-1)K(x,N-1,M-1)$$
$$+dxY(N-1,M-1)K(x-1,N-1,M-1)\Bigg], \tag{70}$$

which, in the thermodynamic limit, reduces to the results identical to the thermodynamic behaviors of the second conditional probability $\mathrm{Prob}(N-x-k-1=\mathrm{car}|N-k=\mathrm{truck})$ $(1 \leqslant k \leqslant N-x-2)$.

V. DENSITY-DENSITY CORRELATION FUNCTION

Here we calculate the two-point equal time correlation function of the car density using the cyclically invariant average (68). Both BSU and FSU schemes are considered simultaneously. In terms of the MPS, the density-density correlation function becomes ($x_1 < x_2$ is assumed)

$$\langle\langle n(x_1)n(x_2)\rangle\rangle = \frac{1}{Z(N,M)}\Bigg[\sum_{\rm conf}\mathrm{Tr}(\cdots D\overbrace{\cdots D\underbrace{\cdots}_{x_1}}^{x_2}\hat{E})$$
$$+\sum_{k=0}^{N-x_2-3}\sum_{\rm conf}\mathrm{Tr}(\cdots D\overbrace{\cdots D\underbrace{\cdots}_{x_1}}^{x_2}E\underbrace{\cdots}_{k}\hat{X}_N) + \sum_{\rm conf}\mathrm{Tr}(\cdots D\underbrace{\cdots}_{x_1}E\underbrace{\cdots}_{N-x_2-2}\hat{D})$$
$$+\sum_{k=0}^{x_2-x_1-2}\sum_{\rm conf}\mathrm{Tr}(\cdots D\underbrace{\cdots}_{x_1}E\underbrace{\cdots}_{N-x_2-2}D\underbrace{\cdots}_{k}\hat{X}_N) + \sum_{\rm conf}\mathrm{Tr}(\underbrace{\cdots}_{x_1}E\underbrace{\cdots}_{N-x_2-2}D\cdots\hat{D})$$
$$+\sum_{k=0}^{x_1-1}\sum_{\rm conf}\mathrm{Tr}(\cdots E\underbrace{\overbrace{\cdots}^{N-x_2-2}D\cdots}_{N-x_1-2}D\underbrace{\cdots}_{k}\hat{X}_N)\Bigg]. \tag{71}$$

After some algebra, it can be verified that

$$\langle\langle n(x_1)n(x_2)\rangle\rangle=\langle\langle n(x_2)\rangle\rangle-f(x_1), \tag{72}$$

where $f(x_1)$ is given by

$$\begin{aligned}f(x_1)=&\frac{1}{Z(N,M)}\Bigg\{\frac{e}{\tilde{\beta}^M}C_{N-3}^{M-1}\\&+dY(N-1,M-1)[1-K(x_1,N-1,M-1)]\\&+\frac{1}{\tilde{\beta}}NY(N-1,M-1)[1-K(x_1,N-1,M-1)]\\&+\frac{d}{\tilde{\beta}}(N-x_1-3)Y(N-2,M-2)\\&\times[1-K(x_1,N-2,M-2)]+\frac{d}{\tilde{\beta}}x_1Y(N-2,M-2)\\&\times[1-K(x_1-1,N-2,M-2)]\Bigg\}.\end{aligned} \tag{73}$$

The connected part of the two-point correlation function, $\langle\langle n(x_1)n(x_2)\rangle\rangle_C=\langle\langle n(x_1)n(x_2)\rangle\rangle-\langle\langle n(x_1)\rangle\rangle\langle\langle n(x_2)\rangle\rangle$, can be used to estimate the degree of correlation. In the thermodynamic limit, one finds

$$\langle\langle n(x_1)n(x_2)\rangle\rangle_C=[\langle\langle n(x_2)\rangle\rangle-\kappa][1-\langle\langle n(x_1)\rangle\rangle], \tag{74}$$

where $\kappa=\min(n,1/\tilde{\beta})$. In the high density phase, the connected part has a nonvanishing value only when both x_1 and x_2 are within the region $[Nl-\sqrt{N}\Delta,Nl+\sqrt{N}\Delta]$ where $\Delta=\sqrt{2\tilde{\beta}(1-n)/(\tilde{\beta}-1)}$. In the low density phase, the connected part has a nonvanishing value only when $x_1<x_2\sim\xi$. Thus one concludes that the correlation develops only in the region where the density variation occurs, which is identical to the conclusion in the RSU scheme [11].

VI. TWO TRUCKS AND BOUND STATE

In this section we examine the system with two trucks and M cars. We determine the probability $\Omega(R)$ that the distance between the trucks is R $[0\leqslant R\leqslant(N-1)/2]$. In the RSU scheme, the cyclic invariance of the MPS allows one to set one of the trucks at the site N, and one obtains

$$\Omega_{\mathrm{RSU}}(R)\sim\sum_{\mathrm{conf}}\mathrm{Tr}(\cdots E\underbrace{\cdots}_{R}E)$$

where the sum runs over all configurations with M cars and two trucks. Its thermodynamic limit is examined in Ref. [11]. In the low density phase $n\beta\leqslant1$, it becomes

$$\Omega_{\mathrm{RSU}}(R)\sim1+\frac{n(1-n)(\alpha+\beta-1)(\alpha-1)}{(1-n+\alpha n)^2}(n\beta)^R, \tag{75}$$

which is maximal at $R=0$ and decays exponentially with the same length scale $\xi_{\mathrm{RSU}}=|\ln(n\beta)|^{-1}$ as in the density profile. In the high density phase $n\beta\geqslant1$, the probability decreases linearly with R for $0\leqslant R\leqslant Nr_{\mathrm{RSU}}$, where $r_{\mathrm{RSU}}=\min(l_{\mathrm{RSU}},1-l_{\mathrm{RSU}})$ $(r_{\mathrm{RSU}}<\frac{1}{2})$, and remains constant for $Nr_{\mathrm{RSU}}\leqslant R\leqslant(N-1)/2$. The relative ratios are

$$\text{for}\quad r_{\mathrm{RSU}}=l_{\mathrm{RSU}},\quad\frac{\Omega_{\mathrm{RSU}}(Nr_{\mathrm{RSU}})}{\Omega_{\mathrm{RSU}}(0)}=1-\frac{(\alpha-1)(\beta-1)}{\alpha\beta} \tag{76}$$

and

$$\text{for}\quad r_{\mathrm{RSU}}=1-l_{\mathrm{RSU}},$$

$$\frac{\Omega_{\mathrm{RSU}}(Nr_{\mathrm{RSU}})}{\Omega_{\mathrm{RSU}}(0)}=1-\frac{(\alpha-1)(\beta-1)}{\alpha\beta}\frac{r_{\mathrm{RSU}}}{1-r_{\mathrm{RSU}}}. \tag{77}$$

One can interpret this as the formation of a weakly bound state between the two trucks.

Now we consider $\Omega(R)$ in the BSU and FSU schemes. In terms of the matrix products, $\Omega(R)$ in both schemes can be expressed (up to a proper normalization constant) as

$$\begin{aligned}&\sum_{k=0}^{N-R-3}\sum_{\mathrm{conf}}\mathrm{Tr}(\cdots E\underbrace{\cdots}_{R}E\underbrace{\cdots}_{k}\hat{X}_N)+\sum_{\mathrm{conf}}\mathrm{Tr}(\underbrace{\cdots}_{N-R-2}E\underbrace{\cdots}_{R}\hat{E})\\&+\sum_{k=0}^{R-1}\sum_{\mathrm{conf}}\mathrm{Tr}(\cdots E\underbrace{\cdots}_{N-R-2}E\underbrace{\cdots}_{k}\hat{X}_N)+\sum_{\mathrm{conf}}\mathrm{Tr}(\underbrace{\cdots}_{R}E\underbrace{\cdots}_{N-R-2}\hat{E}).\end{aligned} \tag{78}$$

It is useful to introduce a quantity $W(N,M,R)$, which is defined by

$$W(N,M,R)=\sum_{\mathrm{conf}}\mathrm{Tr}(\cdots E\underbrace{\cdots}_{R}E). \tag{79}$$

Then, using the relation $\hat{A}=A$, $\hat{D}=D+d$, and $\hat{E}=E+e$, and the cyclic invariance of the trace, expression (78) can be written in terms of $W(N,M,R)$ as follows:

$$NW(N,M,R) + d(N-R-2)W(N-1,M-1,R) + dRW(N-1,M-1,R-1)+2eY(N-1,M). \tag{80}$$

The thermodynamic limit can be investigated in a simple way using the fact that in the RSU scheme, $W(N,M,R)$ is identical to $\Omega_{\mathrm{RSU}}(R)$ up to a normalization factor. Then through the renormalization of α and β, its R dependence in the BSU and FSU schemes can be obtained. Also, the last term in expression (80) is negligible compared to the first three terms. In the high density phase, one then finds that $W(N,M,R)$, $W(N-1,M-1,R)$, and $W(N-1,M-1,R-1)$ are all proportional to each other, and thus $\Omega(R)$ can be obtained from $\Omega_{\mathrm{RSU}}(R)$ through the renormalization of α and β. In the low density phase, the R dependence of $W(N,M,R)$ appears only for $R\sim\xi$. It is then sufficient to examine the case $R\sim\xi\ll N$, where the first and second terms are dominant and give the same R dependence. Hence $\Omega(R)$ can be again obtained from $\Omega_{\mathrm{RSU}}(R)$ by replacing α and β by their renormalized values.

VII. CONCLUDING REMARKS

We have investigated the characteristics of an exactly solvable two-way traffic model with ordered sequential updates, and observed both qualitative and quantitative differences in the properties of model from the results obtained with the random sequential update [11]. Our approach is based on the so-called matrix product formalism which allows analytic solutions. In the OSU schemes, the choice of a particular site as a starting point of the update breaks the translational invariance of the steady state measure, which is also evident in the form of the MPS. Thus an averaging over the different choices of the update starting point is necessary to restore the cyclic invariance to the system [see Eq. (68) and the following discussion]. Performing the cyclically invariant averaging, some characteristics in the thermodynamic limit, such as density profile of cars, density-density correlation function, and truck-truck distance distribution $\Omega(R)$, are obtained, and it is found that the difference in the updating schemes can be taken into account simply by the proper renormalization of the parameters α and β. However this is not the case with average velocities. Changing the update scheme affects velocities in a more complicated manner and the renormalization of the parameters is not sufficient to account for different behaviors of $\langle v_{\mathrm{car}}\rangle$ and $\langle v_{\mathrm{truck}}\rangle$ in different updating schemes. Especially the dependence of $\langle v_{\mathrm{car}}\rangle$ and $\langle v_{\mathrm{truck}}\rangle$ on the road narrowness $1-(1/\beta)$ can vary qualitatively depending on the updating schemes. Behaviors of $\langle v_{\mathrm{car}}\rangle$ and $\langle v_{\mathrm{truck}}\rangle$ in the FSU scheme (that is, when the update direction is parallel with the movement direction of the majority of vehicles) have more resemblance to the RSU results than those in the BSU scheme. In the FSU scheme, however, one observes a shift in the value of the critical narrowness from $1-n$ to $[(1-\eta)(1-n)]/[1-\eta(1-n)]$, which can be considerable if $\eta\approx 1$.

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APPENDIX: DERIVATION OF EXPRESSION (27)

Here we derive expression (27) for the average current of cars in the BSU scheme. Similar procedures can be used to obtain average currents of the truck and also the average currents in the FSU scheme. The starting point is the continuity equation, which in discrete time dynamics takes the form

$$\langle n_k^{\mathrm{car}}\rangle_{j+1}-\langle n_k^{\mathrm{car}}\rangle_j=\langle J_{k-1}^{\mathrm{car}}\rangle_j-\langle J_k^{\mathrm{car}}\rangle_j, \tag{A1}$$

where $\langle\cdots\rangle_j$ represents the average at the time step j. In terms of the initial state of the system $|P,0\rangle$, the left hand side becomes

$$\langle s|n_k^{\mathrm{car}}T_{\leftarrow}T_{\leftarrow}^j|P,0\rangle-\langle s|n_k^{\mathrm{car}}T_{\leftarrow}^j|P,0\rangle, \tag{A2}$$

where the bra vector $\langle s|$ is defined by

$$\langle s|=\sum_{\{\tau\}}\langle\tau_1|\otimes\langle\tau_2|\otimes\cdots\otimes\langle\tau_N|. \tag{A3}$$

The conservation of the probability ensures that $\langle s|T_{\leftarrow}=\langle s|$ [3], which then allows one to rewrite Eq. (A2) as $\langle s|[n_k^{\mathrm{car}},T_{\leftarrow}]T_{\leftarrow}^j|P,0\rangle$. Next we evaluate the commutator $[n_k^{\mathrm{car}},T_{\leftarrow}]$. Using the relation

$$n_k^{\mathrm{car}}=\underbrace{\mathbf{1}\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1}}_{k-1}\otimes|1\rangle\langle 1|\otimes\underbrace{\mathbf{1}\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1}}_{N-k},$$

one finds

$$T_{N,1}\cdots T_{k-2,k-1}\left(\eta E_{k-1}^{0,1}E_k^{1,0}+\frac{\eta}{\beta}E_{k-1}^{2,1}E_k^{1,2}\right)T_{k,k+1}\cdots T_{N-1,N}-T_{N,1}\cdots T_{k-1,k}\left(\eta E_k^{0,1}E_{k+1}^{1,0}+\frac{\eta}{\beta}E_k^{2,1}E_{k+1}^{1,2}\right)T_{k+1,k+2}\cdots T_{N-1,N} \tag{A4}$$

where

$$E_k^{i,j} = \underbrace{\mathbf{1} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{k-1} \otimes |i\rangle\langle j| \otimes \underbrace{\mathbf{1} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{N-k} .$$

Comparison with the right-hand side of Eq. (A1) (note that the two currents have different subscripts $k-1$ and k) shows that each term in Eq. (A4) should lead to expressions for $\langle J_{k-1}^{\text{car}}\rangle_j$ and $\langle J_k^{\text{car}}\rangle_j$, respectively. Then by taking the limit $j\to\infty$ and using expression (16) for the steady state $|P_s\rangle = \lim_{j\to\infty} T_{\leftarrow}^{j}|P,0\rangle$, one finds

$$\langle J_k^{\mathbf{car}}\rangle = \frac{1}{Z(N,M)}\langle s| \left(\eta E_k^{0,1} E_{k+1}^{1,0} + \frac{\eta}{\beta} E_k^{2,1} E_{k+1}^{1,2}\right) \times \mathrm{Tr}\left(\underbrace{|U\rangle \otimes |U\rangle \otimes \cdots |U\rangle}_{k} \otimes |\hat{U}\rangle \otimes |U\rangle \otimes \cdots \otimes |U\rangle\right) \quad \text{(A5)}$$

where the effects of $T_{l,l+1}$ on $|P_s\rangle$ and $\langle s|$ have been taken into account. Finally we use

$$|i\rangle\langle j|(A_0|0\rangle + A_1|1\rangle + A_2|2\rangle) = A_j|i\rangle,$$

which yields expression (27).

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